METRIC AND TOPOLOGICAL SPACES: EXAM 2020

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Problem 1 ($6+3+6=15\%$). Decide if the function $\mathrm{d}(x,y)=\{0$ if $x=y$, $x+y$ if $x\neq y\}$ is or is not a metric on the set $\mathbb{N}_{\geqslant 1}$, on $\mathbb{N}_{\geqslant 0}$, on $\mathbb{N}_{\geqslant 0}\cup\{-1\}$, on $\mathbb{R}_{\geqslant 0}$. If 'yes' at least once, then draw the respective open disks $B_{r=10}(x_0=2)$ and $B_{r=2}(x_0=10)$.

Problem 2 (15%). Let $(\mathcal{X},\mathrm{d}_{\mathcal{X}})$ and $(\mathcal{Y},\mathrm{d}_{\mathcal{Y}})$ be metric spaces. Prove that a map $f\colon\mathcal{X}\to\mathcal{Y}$ is continuous if and only if $f(\overline{A})\subseteq\overline{f(A)}$ for all subsets $A\subseteq\mathcal{X}$ (the bar denotes closure).

Problem 3 (12%). In a space $\mathcal{X}$, its subset V is closed if and only if it contains own boundary: $\partial V\subseteq V$. (prove)

Problem 4 (13%). Give an example of a sequence of open connected subsets $U_n\subseteq\mathbb{E}^2$ of the plane such that $U_n\supseteq U_{n+1}$ for each $n\in\mathbb{N}$ but the intersection $\bigcap_{n=1}^{+\infty}U_n$ is not connected.
(NB: $\varnothing$ is connected)

Problem 5 ($5+5+10=20\%$). Let $\mathcal{X}$ be a Hausdorff space.
(a) In $\mathcal{X}$, every compact is closed. (prove)
(b) For any point $x\in\mathcal{X}$, the intersection of all open subsets of $\mathcal{X}$ containing x is the singleton set $\{x\}$. (prove)
(c) Give an example of topological space $\mathcal{X}$ which is not Hausdorff, its topology is not co-finite, still (b) holds.

Problem 6 (25%). **Theorem:** A metric space $\mathcal{X}$ is complete. $\Longleftrightarrow$ Any nested sequence of closed disks $V_n:=B_{\overline{r_n}}(x_n)=\{y\in\mathcal{X}\mid\mathrm{d}(x_n,y)\leqslant r_n\}\supseteq V_{n+1}$ such that their radii $r_n\to 0$ as $n\to\infty$ has a non-empty intersection. Prove $\Longleftarrow$.

Date: March 31, 2020 (08:30–12:00). Good luck & take care ! Reserve: `guest16479-student.rug.nl`.